# Asymptotic Expansions of the Stable Distribution

Karina Arias-Calluari[1],* Fernando Alonso-Marroquin [1,2], and Michael Harre [1]
[1]*School of Civil Engineering, University of Sydney, Sydney NSW 2006 and*
[2]*Computational Physics IfB, ETH Zurich, CH-8093, Zurich, Switzerland*
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The stable distribution is the attractor of distributions which hold power laws with infinite variance. This function does not have an explicit expression and no uniform solution has been proposed yet. This paper presents a uniform analytical approximation for the stable distribution based on matching power series expansions. For this solution, the trans-stable function is defined as an auxiliary function to evaluate the stable distribution. The trans-stable function removes the numerical issues of the calculations of the stable function and it presents the same asymptotic behaviour as the stable distribution function. To obtain the uniform analytical approximation two types of approximations are developed. The first one is called "inner solution" and it is an asymptotic expansion around $x = 0$. The second refers to "outer solution" and it is a series expansion for $x \to \infty$. Then, the uniform solution is proposed as a result of an asymptotic matching between inner and outer solutions. Finally, the results of analytical approximation were compared to the numerical results of the stable distribution function, making this uniform solution valid to be applied as an analytical approximation.

## I. INTRODUCTION

A wide range of natural and social phenomena exhibit a power-law in the probability distribution of large events. These tails are characterized by the asymptotic relation $f(x) \sim 1/x^{1+\alpha}$, where $x$ is the size of the events.

For $0 < \alpha \leq 1$, the distribution has an indefinite mean value. Examples that fall in this range are the affected areas of bushfires [1], wildfires and rockfalls [2], intensity and duration of rains [3], neuron firing rates and neural avalanches [4, 5], call holding times and packed inter-arrival data of the internet [6], war intensities [7] and personal identity losses on internet [8].

For $1 < \alpha \leq 2$, the distribution has a defined mean value but still exihibits an infinite variance. A different range of examples have been found, such as landslide areas triggered by earthquakes [7, 9], heavy rains and snowmelts [9], a probability distribution of monthly river flows [10], daily returns in currency exchange rates [11], the effects of networks in price returns [12], daily returns of Dow Jones index [11] and benefits and returns of top Hollywood movies [13].

Section 35 of the book by Gnedenko and Kolmogorov [14] shows that the normal distribution is an "attractor" of distributions with finite variances. On the other hand, the attractor of distribution holding power laws with infinite variances corresponds to the more general "stable law".

The fundamental concept of attractors is formulated as follows: If a normalized sum of a set of independent, identically distributed random variables $\{X_1, X_2, X_3 .... X_N\}$ satisfies:

$$\lim_{N\to\infty} \frac{1}{\sigma_N}\left(\sum_{i=1}^{N} X_i - \mu_N\right) = X. \tag{1}$$

Then $X$ belongs to the stable law. The coefficients $\mu_N$ and $\sigma_N$ represent the centering and normalizing values respectively [14].

The Gnedenko-Kolmogorov theorem is a generalization of the classical central limit theorem which states that normalized sum of independent random variables with finite variance in Eq. (1) converges to a variable that is normally distributed [14, 15]. This is the case of distributions with power-law tails with finite variance $\alpha \geq 2$. The normalized coefficient is $\sigma_N = \sqrt{N}$ and the centering coefficient is $\mu_N = NE[X]$, where $N$ represents the length of the sum and $E[X]$ refers to expected value [16, 17]. On the other hand, for independent random variables holding power laws with infinite variance[1] $0 < \alpha < 2$, Uchaikin and Zorotalev [17, 18] show that $X$ in Eq. (1) follows a symmetric stable law if the normalization coefficient is $\sigma_N = N^{1/\alpha}$ and the centering coefficient is $\mu_N = 0$ for $\alpha \leq 1$ or $\mu_N = NE[X]$ for $\alpha > 1$.

The stable distribution function is given by an integral that has analytical solution only for few cases —normal and Cauchy distributions— in the remaining cases, it does not have a closed-form expression. The numerical solution of the stable distribution has numerical oscillations specifically in the tails. For some cases it displays apparent discontinuity in logarithmic scale plots because of negative values obtained from the numerical solution [20]. Because of that, it can be said that the numerical solution of the stable distribution is not reliable, because the probability density function must be always positive.

Analytical expressions in terms of power series were presented by different authors. Feller [19], Elliot [21] and Zolotarev [18] used power series to obtain converging algorithms of the stable distribution function in two ranges, the first one for $\alpha < 1$ and the second for $\alpha > 1$ for symmetric distributions. However, some of the proposed series do not converge to the stable function, and some

* School of Civil Engineering; kari0293@uni.sydney.edu.au

[1] Note: Infinite variance is observed for $0 < \alpha < 2$. This characteristic occurs for $0 < \alpha \leq 1$, as a consequence of not having a well-defined expected value $E[X]$. For $1 < \alpha < 2$, the integral in the variance definition diverges [17–19].

of them are only applicable for extreme values $x \to 0$ or $x \to \infty$. Mantegna [22] presented a quite similar solution to Elliot [21] but his algorithm is only valid when $x \to \infty$ and $0.75 < \alpha \leq 1.95$. Nolan [23] presented his algorithms considering asymmetric distributions for large events $x \to \infty$ focusing on tail behaviour only. Thus, the stable distribution function does not have an explicit expression [24, 25] and no uniform solution of the stable distribution has been proposed until now [18, 19, 23].

Due to the absence of an explicit expresion, numerical solutions were developed to evaluate the stable distribution function by using numerical recursive quadrature methods [26–28]. Nolan [26, 29] develops a numerical solution for the estimation of stable parameters through a maximun likelihood method for each data set of $x$. However, Nolan's method converges only for $\alpha > 0.4$ and the convergence to the stable distribution function seems to be not accurate enough. Despite this fact, Nolan's method constitutes an important method that is still being used [27].

Apart from the numerical issues in the evaluation of the stable distribution, some authors have pointed out its infinite variance as a drawback [30–32]. To avoid the infinite variance of the stable distribution function, several truncations are proposed. The truncations make the variance finite, consequently the distribution function of the sum of independent random variables converges to the normal distribution due to the central limit theorem for a large $N$ value. Nevertheless, a time series in the real phenomena can exhibit infinite variance, one of these cases is the variance of price fluctuations on stock markets, which increases when the time frame is enlarged [33, 34].

The aim of this paper is to formulate a uniform analytical approximation for the stable distribution function based on a series expansion. To achieve this aim we propose several regularizations of the inner and outer series expansions to ensure convergence. This will be an important tool to get the most accurate approximation reducing numerical errors (oscillations) when the stable function is evaluated.

This paper is divided in two parts. The first part introduces the stable distribution and the trans-stable function. They are defined by Fourier transformations in sections II and III respectively. The trans-stable function is shown to be identical to the stable distribution for $\alpha < 1$ and it has the same asymptotic behaviour for $\alpha > 1$ for large events. The second part refers to section IV and it deals with the closest form— analytical approximations— of the stable distribution. For this purpose, two types of approximations are developed. The first approximation refers to the inner expansion that converges asymptotically to the stable distribution as $x \to 0$. The second approximation refers to the outer expansion that converges asymptotically as $x \to \infty$. For the outer expansion two cases are presented, one is obtained from the stable function in subsection IV B and the second one from trans-stable function in subsection IV C. Finally, the uniform solution in section V is proposed as a result of matching the inner and the outer solution. The analytical equation of uniform solution proposed in this paper gives an approximated solution of the stable distribution function over the range $-\infty < x < \infty$.

## II. STABLE DISTRIBUTION FUNCTION

The stable distribution is presented in Section 34 of the Gnedenko-Kolmogorov book [14] as the Fourier transformation:

$$f(x;\alpha,\beta,\sigma,\mu) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \varphi(t;\alpha,\beta,\sigma,\mu) e^{ixt} dt, \qquad (2)$$

where $\varphi(t)$ is defined as the characteristic function,

$$\varphi(t;\alpha,\beta,\sigma,\mu) = e^{(it\mu - |\sigma t|^{\alpha}(1 - i\beta sgn(t)\Phi))}. \qquad (3)$$

The four parameters involved are: the stability parameter $\alpha \in \langle 0 \ 2]$, the skewness parameter $\beta \in [-1 \ +1]$, the scale parameter $\sigma \in (0 \ +\infty)$, and the location parameter $\mu \in (-\infty + \infty)$. The parameter $\alpha$ constitutes the characteristic exponent of the asymptotic power-law in the tails and it determines whether the mean value and the variance exist. The stable distribution with $0 < \alpha \leq 1$ does not have a mean value and it has a define variance only for $\alpha = 2$ [35].

The function $sgn\,(t)$ represents the sign of $t$ and the function $\Phi$ is defined as:

$$\Phi = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \alpha \neq 1, \\ -\frac{2}{\pi}\log|t| & \alpha = 1. \end{cases} \qquad (4)$$

The stable distribution is the family of all attractors of normalized sums of independent and identically distributed random variables. The most well-known stable distribution functions are the Cauchy distribution with $\alpha = 1$ and the normal distribution function with $\alpha = 2$. Both functions have $\beta = 0$, which means they are symmetric distributions about their mean [18].

In this paper we will focus on symmetric distributions ($\beta = 0$). For this case the stable distribution can be normalized as follows:

$$f(x;\alpha,\beta=0,\sigma,\mu) = Re\left\{S\left(\frac{x-\mu}{\sigma},\alpha\right)\right\}, \qquad (5)$$

where the general distribution function is given by the following equation:

$$S(x;\alpha) = \frac{1}{\pi}\int\limits_{0}^{\infty} e^{-t^{\alpha}} e^{ixt} dt. \qquad (6)$$

The real part of this function corresponds to the normalized stable distribution,

$$s(x;\alpha) = Re(S(x;\alpha)).$$

Consequently,

$$s(x;\alpha) = \frac{1}{\pi}\int_0^\infty e^{-t^\alpha}\cos(tx)dt. \quad (7)$$

## III. TRANS-STABLE FUNCTION

Zolotarev (1986) used the term "trans-stable" to refer to a power series expansion that converges to the stable distribution for $0 < \alpha < 1$ only [18]. In this paper, trans-stable is the function which one of its solutions originates Zolotarev series when the series expansions are applied around $x \to \infty$. First we define *the complex trans-stable function* in the range of $0 < \alpha < 2$. For $\alpha < 1$, the stable distribution and the trans-stable function are identical. For $\alpha > 1$, the trans-stable function and the stable distribution present the same asymptotic behaviour for $x \to \infty$. Consequently, our trans-stable function can be used to find a numerical aproximation of the stable distribution function for $\alpha > 1$ for large events.

First, *the complex trans-stable function* is defined as an integral over the path $C$ in the complex plane:

$$G_C\left(x;\alpha\right) = \frac{1}{\pi}\int_C I\left(x,z;\alpha\right)dz, \quad (8)$$

where

$$I\left(x,z;\alpha\right) = e^{-z^\alpha}e^{ixz}. \quad (9)$$

The relation of this function to the stable $S(x;\alpha)$ and the trans-stable $T(x;\alpha)$ functions is obtained by choosing a particular path $C$ in the complex plane. Then, the stable distribution and trans-stable function are given by Eq. (10) and (11) respectively:

$$S(x;\alpha) = G_{[0,\infty)}(x;\alpha) = \frac{1}{\pi}\int_0^\infty e^{-t^\alpha}e^{ixt}dt, \quad (10)$$

$$T(x;\alpha) = G_{[0,i\infty)}(x;\alpha) = \frac{1}{\pi}\int_0^{i\infty} e^{-t^\alpha}e^{ixt}dt. \quad (11)$$

First it will be shown that for $0 < \alpha \leq 1$, both stable $S(x;\alpha)$ and trans-stable $T(x;\alpha)$ functions are identical. For $1 < \alpha < 2$ it will be demonstrated that both functions exhibit the same asymptotic behaviour when $x \to \infty$.

This demonstration is based on the evaluation of the *complex trans-stable integral* Eq. (8) using polar representation for $\alpha \leq 1$ and rectangular representation for $\alpha > 1$ on the complex integrand. The demonstrations are presented in the following subsections.

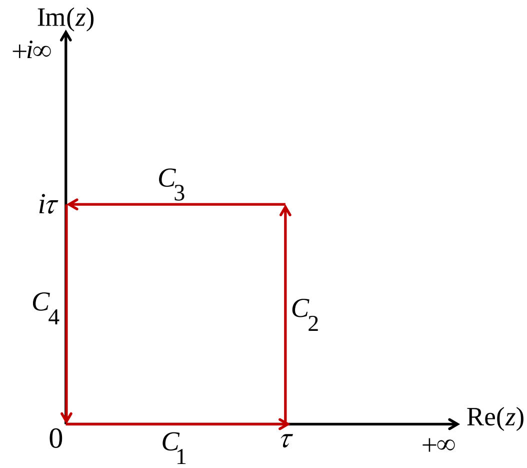


**Figure 1**. *Contour integration for* Eq. (8)

### 1. *For* $0 < \alpha \leq 1$

Here we will show that for $0 < \alpha \leq 1$ the stable and trans-stable functions are identical. This demonstration will be done by considering the closed contour shown in Figure 1. Since the complex function in Eq. (9) is analytical over the complex plane, the integral over the closed contour Eq. (8) is zero,

$$\oint I\left(x,z;\alpha\right)dz = 0. \quad (12)$$

Let us take the contour in Figure 1 that can be divided into four straight paths so that:

$$\sum_1^4 \int_{C_i} I\left(x,z;\alpha\right)dz = 0. \quad (13)$$

Now, we will take the limit when $\tau \to \infty$ in Figure 1. Using Eq. (10,11,13) the following equation is obtained:

$$S(x;\alpha) - T(x;\alpha) = -\lim_{\tau\to\infty}\sum_{i=2}^{3}\int_{Ci} I\left(x,z;\alpha\right)dz. \quad (14)$$

To evaluate the right hand side in Eq. (14) it is convenient to use the polar representation of the complex number $z = re^{i\theta}$ and express Eq. (9) in polar coordinates:

$$I\left(x,z;\alpha\right) = e^{g(x,r,\theta;\alpha)+ih(x,r,\theta;\alpha)}, \quad (15)$$

$$g(x,r,\theta;\alpha) = -r^\alpha\cos\left(\theta\alpha\right) - rx\sin\theta, \quad (16)$$

$$h(x,r,\theta;\alpha) = -r^\alpha\sin\left(\theta\alpha\right) + rx\cos\theta. \quad (17)$$

It will be adopted the nomenclature of signal theory, where the polar notation separates the effects of instantaneous amplitude $|I| = e^g$ and its instantaneous phase $h$ of a complex function [36]. Consequently, $g(x, r, \theta; \alpha)$ represents the *attenuation factor* and $h(x, r, \theta; \alpha)$ represents the *oscillation factor*.

Now let us notice that $\lim_{r\to\infty} g(x, r, \theta; \alpha) = -\infty$ for $0 < \alpha \leq 1$ at any value of $x$. This statement is based on the fact that $cos(\theta\alpha) \geq 0$ in the first quadrant for $\alpha \leq 1$. Consequently, $\lim_{r\to\infty} I\,(x, z; \alpha) = 0$ so that the integral of the right side of the Eq. (14) vanishes at $\tau \to \infty$, therefore:

$$S(x;\alpha) = T(x;\alpha) \quad \text{if} \quad 0 < \alpha \leq 1. \tag{18}$$

So, the Eq. (18) will allow to use the trans-stable function $T(x;\alpha)$ instead of the stable distribution function $S(x;\alpha)$ for $0 < \alpha \leq 1$ in the numerical integration. This is with the aim to remove numerical oscillation, specifically in the tails. It is noticeable that the integration of the trans-stable function $T(x;\alpha)$ in Eq. (11)is performed over the imaginary axis. Applying the following change of variable $t \to -it$ (formally done by defining $u = -it$ so that $du = -idt$ and later replacing the dummy variable $u$ by $t$ inside the integral), the trans-stable function is converted into a Laplace transformation. Consequently, the integration is performed over the real axis. The Fourier and Laplace representations for $T(x;\alpha)$ are shown in Eq. (19),

$$T(x;\alpha) = \frac{1}{\pi} \int_0^{i\infty} e^{-t^\alpha} e^{ixt} dt = \frac{1}{\pi} \int_0^{\infty} e^{-(it)^\alpha} e^{-xt} i dt. \tag{19}$$

Figure 2 compares the Fourier representation of the stable distribution function $S(x;\alpha)$ and the Laplace representation of trans-stable function $T(x;\alpha)$. The integration is performed using a recursive adaptive Simpson quadrature method [37]. It is evident that the Laplace representation removes the oscillations of the Fourier representation of the stable distribution for $\alpha < 1$.

It is important to add that the stable distribution function and trans-stable function hold the same value for their Fourier and Laplace transform representations. The difference between each transform representation is the axis in which each function is integrated. The expressions are shown in Table I.

| | Stable distribution $S(x;\alpha)$ | Trans-stable function $T(x;\alpha)$ |
|---|---|---|
| Fourier transform | $\frac{1}{\pi}\int_0^{\infty} e^{-t^\alpha} e^{ixt} dt$ | $\frac{1}{\pi}\int_0^{i\infty} e^{-t^\alpha} e^{ixt} dt$ |
| Laplace transform | $\frac{1}{\pi}\int_0^{-i\infty} e^{-(it)^\alpha} e^{-xt} i dt$ | $\frac{1}{\pi}\int_0^{\infty} e^{-(it)^\alpha} e^{-xt} i dt$ |

**Table I**. *Summary of Fourier and Laplace representations for the stable and the trans-stable functions*

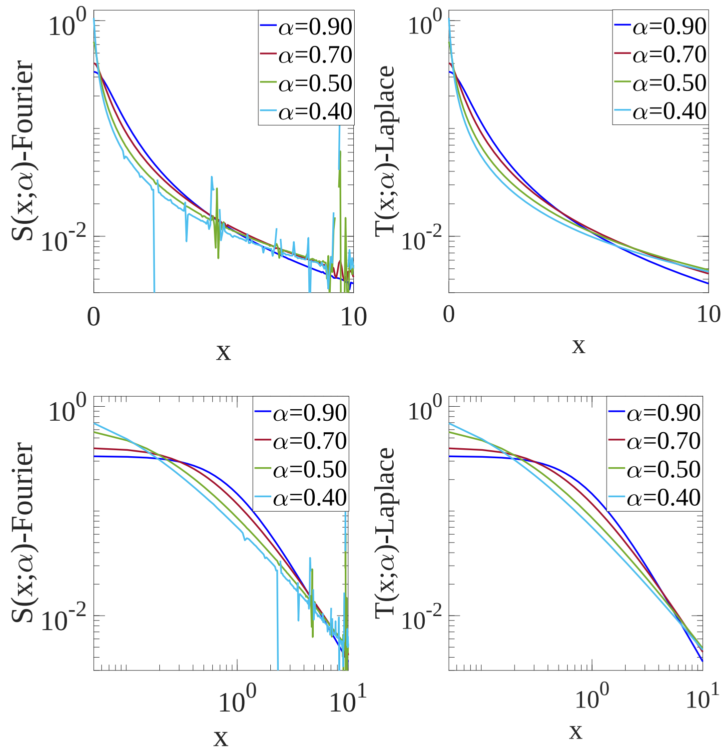


**Figure 2**. *Comparison of numerical integration* $0 < \alpha < 1$ *between Fourier and Laplace transform of the stable* $S(x;\alpha)$ *and the trans-stable* $T(x;\alpha)$ *functions using recursive adaptive Simpson quadrature method [37]. The absolute error tolerance of the method is* $\xi = 3.5 \times 10^{-8}$. *Top plots are shown in semi-logarithmic scale and in logarithmic scale.*

### 2. *For* $1 < \alpha < 2$

Here we will shown that for $1 < \alpha < 2$ the stable and trans-stable functions have the same asymptotic behaviour on large events if the integrals are appropriately truncated.

Let us recall Eq. (16) for the *attenuation factor*,

$$g(x, r, \theta; \alpha) = -r^\alpha \cos(\theta\alpha) - rx \sin\theta.$$

In the previous section, it was shown that $cos(\theta\alpha)$ is always positive in the first quadrant of the complex plane if $0 < \alpha \leq 1$. Otherwise, if $\alpha > 1$, then $cos(\theta\alpha) < 0$ when $\theta = \pi/2$. Consequently, $\lim_{r\to\infty} I(x, r, \theta; \alpha) = \infty$ in this range, so that the right hand side of Eq. (14) can not be neglected. Therefore $S(x) \neq T(x)$ if $\alpha > 1$.

We can find an approximation between these two functions if the $\tau$ value in the contour of Figure 1 is kept large but finite ($\tau < \infty$). Thus, Eq. (13) becomes:

$$S(x;\alpha,\tau) - T(x;\alpha,\tau) = -\sum_{i=2}^{3} \int_{Si} I\,(x, z; \alpha)\, dz, \tag{20}$$

where $S(x;\alpha,\tau)$ and $T(x;\alpha,\tau)$ are the truncated integrals in Eqs. (10) and (11) respectively:

$$S(x;\alpha,\tau) = \frac{1}{\pi}\int_0^{\tau} e^{-t^{\alpha}} e^{ixt} dt, \tag{21}$$

$$T(x;\alpha,\tau) = \frac{1}{\pi}\int_0^{i\tau} e^{-t^{\alpha}} e^{ixt} dt. \tag{22}$$

Now, the right hand of Eq. (20) can be evaluated in the limit where $x \to \infty$. First, notice that in the contour of integration in Figure 1 the magnitude of $r$ is bounded by the condition $0 < r < \sqrt{2}\,\tau$ and $sin(\theta) > 0$ in the first quadrant, thus:

$$\lim_{x\to\infty} g(x,r,\theta;\alpha) = -\infty.$$

Consequently, $\lim_{x\to\infty} I\,(x,z;\alpha) = 0$ so that the integral on the right of Eq. (20) vanishes at $x \to \infty$. Therefore, the asymptotic behavior is obtained for $1 < \alpha < 2$,

$$S(x;\alpha,\tau) \sim T(x;\alpha,\tau) \quad \text{as} \quad x \to \infty. \tag{23}$$

This demostrates that both functions are asymptotically equivalent when the integrals are truncated.

The next step is to find the truncantion value $\tau$ that leads to the best approximation of these functions. The value of $\tau$ should be chosen to minimize the truncation error and at the same time to make the domain of integration as small as possible. With this aim, the trans-stable function $T(x;\alpha,\tau)$ in Eq. (11) is expressed in its Laplace representation by usying the change of variable $t \to -it$. Thus,

$$T(x;\alpha,\tau) = \frac{1}{\pi}\int_0^{\tau} \bar{I}\,(x,t;\alpha)\,dt, \tag{24}$$

where $\bar{I}$ corresponds to Laplace transform integrand shown in Eq. (19) and Table I,

$$\bar{I}\,(x,t;\alpha) = e^{(-it)^{\alpha}} e^{-xt} i. \tag{25}$$

Then, considering Euler's representation for a complex exponential function $e^{i\theta} = cos\,(\theta) + isin\,(\theta)$, the following equations are obtained to express Eq. (25):

$$\bar{I}\,(x,t;\alpha) = e^{\bar{g}(x,t;\alpha)+i\bar{h}(x,t;\alpha)}, \tag{26}$$

$$\bar{g}(x,t;\alpha) = -t^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right) - xt, \tag{27}$$

$$\bar{h}(x,t;\alpha) = -t^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right) + \frac{\pi}{2}. \tag{28}$$

The instantaneous amplitude $|\bar{I}| = e^{\bar{g}}$ will be determined by the *attenuation factor* in Eq. (27). For that reason, an analysis of $\bar{g}(x,t;\alpha)$ was made in Figure 3. The curve $\bar{g}(x,t;\alpha) = 0$ divides two regions, one with exponential growth ($\bar{g} > 0$) and the other with exponential decay ($\bar{g} < 0$).

In Figure 3, two sub-regions can be recognized in $\bar{g} < 0$. The first one, "Zone A" which contains negative $\bar{g}$ values with downward trend $\partial g/\partial t < 0$ that is faster as $x \to \infty$. The second sub-region is "Zone B", it contains smaller negative $\bar{g}$ values that follow an upward trend and $\partial g/\partial t > 0$ displaying an increase behaviour when $x \to 0$. Considering these sub-regions, the truncation $\tau$ in Eq. (24) will depend on $x$ value as follows:

- For $x \to 0$, The integration must avoid zone B. The values of $\bar{g}(x,t;\alpha)$ in this zone lead to an exponential growth due to an upward trend $\partial g/\partial t > 0$, consequently $|\bar{I}| \nrightarrow 0$.
- For $x \to \infty$, the integration should be restricted to zone A. The downward trend $\partial g/\partial t < 0$ leads to obtain $\bar{g}(x,t;\alpha) \to 0$. Consequently, the convergence of $|\bar{I}| \to 0$ occurs faster as $t \to \infty$.

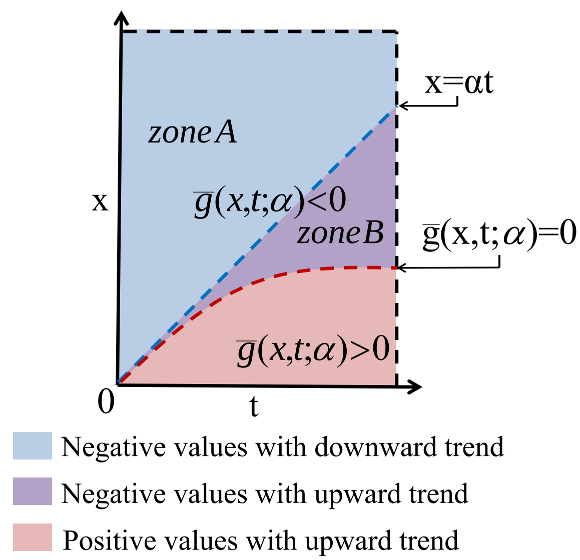


**Figure 3**. *Curve $\bar{g} = 0$ separates two regions with $\bar{g} > 0$ and $\bar{g} < 0$. In the latter region, two zones can be distinguished: zone A with $\partial g/\partial t < 0$ and zone B with $\partial g/\partial t > 0$. The equation $x = \alpha t$ is an estimation of the boundary between zones A and B.*

For $x \to 0$, the cut off $\tau_1$ which avoids most of zone B is defined by $x = \alpha t$. This equation is an estimation of the boundary between zones A and B for all range of $\alpha$ values.

The cut off $\tau_1$ obeys a linear equation and is obtained from the following equations:

$$e^{\bar{g}(x,\tau_1;\alpha)} = |\bar{I}| = \epsilon \quad \text{and} \quad x = \alpha\tau_1, \tag{29}$$

where the tolerance $\epsilon$ represents a negligible instantaneous amplitude $|\bar{I}|$.

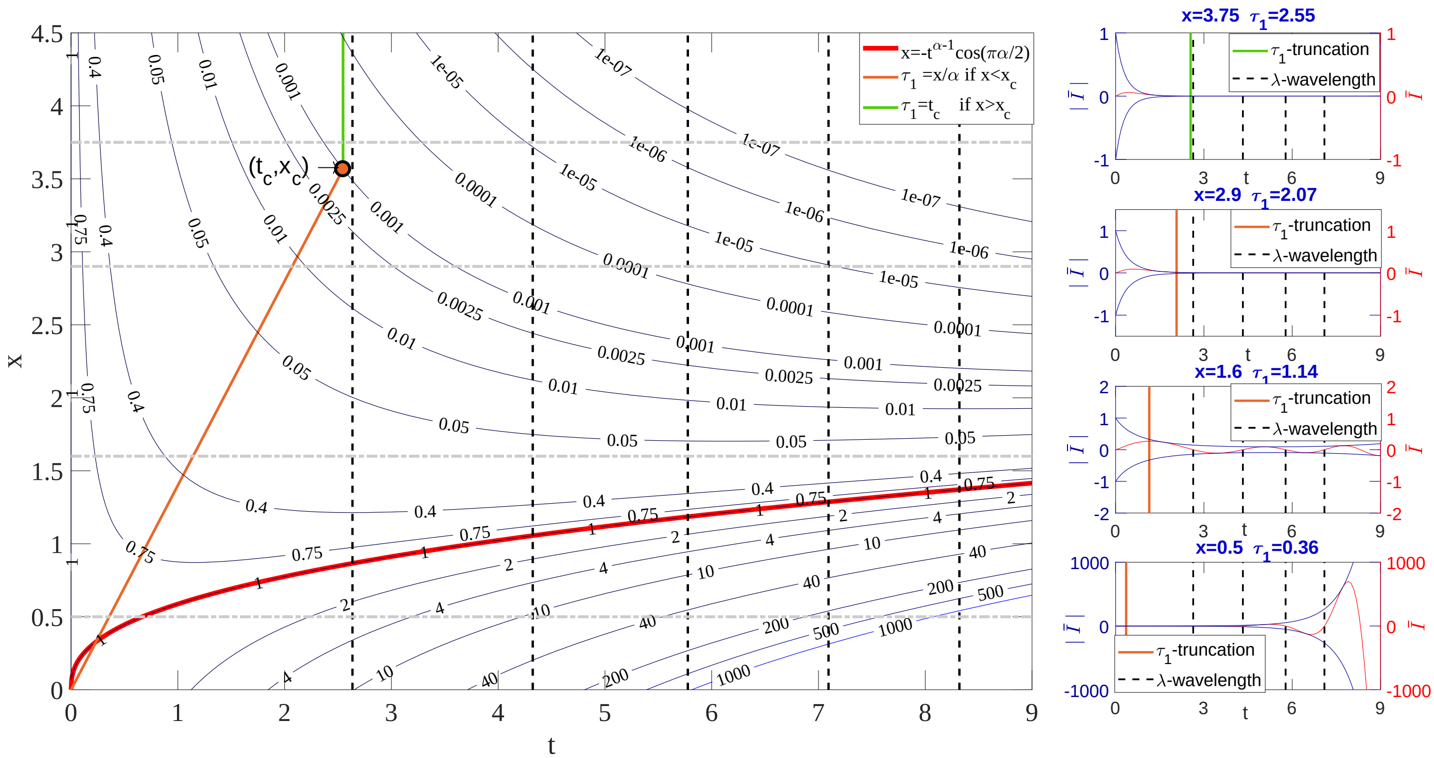


**Figure 4**. *Contour plot of* $e^{\bar{g}}$ *that represents the instantaneous amplitude* $|\bar{I}|$ *for* $\alpha = 1.4$ *and tolerance* $\epsilon = 10^{-3}$ *in Eq. (26) and (27). The red line* $x$ *was drawn by considering* $\bar{g} = 0$ *which represents the limit between the positive and negative values of the attenuation factor* $\bar{g}$*. The truncation* $\tau_1$ *is applied following Eq. (30). Downward trend before* $\tau_1$ *and upward trend after* $\tau_1$ *of* $|\bar{I}|$ *can be observed in the right part of the figure for* $x = 0.5$ *and* $x = 1.6$*. For* $x = 2.9$ *and* $x = 3.8$ *the truncation is made after reaching a small value on the modulation* $\bar{I}$*.*

For $x \to \infty$, the cut off $\tau_1$ will restrict the integration of $\bar{I}$ on a closed interval $[0, t_c]$. This occurs due to a faster downward trend $\partial g/\partial t < 0$. The $t_c$ value represents the point where the instantaneous amplitude can be considered a negligible quantity $|\bar{I}| = \epsilon$. Thus, the cut off $\tau_1$ obeys an equation of a vertical line $\tau_1 = t_c$.

Notice that there are two different definitions for $\tau_1$. Each one corresponds to a particular sub-regions A ($x \to \infty$) or B ($x \to 0$). Consequently, the truncation $\tau_1$ for the trans-stable function is defined by two equations which depend on the $x$ and $\epsilon$ values. These two equation have their intersection point at $(t_c, x_c)$:

$$\tau_1(\epsilon, x) = \begin{cases} t_c(\epsilon) & if \quad x > x_c \\ x/\alpha & if \quad x < x_c \end{cases} \quad for \quad \alpha > 1, \qquad (30)$$

where $t_c(\epsilon)$ and $x_c$ are given by the implicit form of the following equations:

$$\begin{gathered} \alpha {t_c}^2 + {t_c}^\alpha \cos(\pi\alpha/2) + \ln(\epsilon) = 0, \\ x_c = \alpha t_c. \end{gathered} \qquad (31)$$

Figure 4 illustrates the contour plot of the instantaneous amplitude $|\bar{I}|$ for $\alpha = 1.4$. The truncation $\tau_1$ is presented as a cut-off made when a negligible value of instantaneous amplitude is achieved $|\bar{I}| = \epsilon = 10^{-3}$. The point $(x_c, t_c)$ is located at the intersection between the contour line of the given tolerance $\epsilon$ and the equation $\tau_1 = x/\alpha$. The truncation $\tau_1$ avoids zone B which contains negative values for $\bar{g}$ with $\partial g/\partial t > 0$. One can observe that there is an abrupt upward trend in $|\bar{I}|$ for $x \to 0$. So, the truncation $\tau_1$ allows us to make a perfect cut off before this upward trend starts. It is noticeable that with a small tolerance $\epsilon$ the intersection will occur in the rightmost part of the figure, consequently the interval of integration will be wider and a more accurate result can be obtained.

Figure 5 shows how the solutions of trans-stable and stable distribution functions are quite similar after $x_c$ value, which depends on the tolerance $\epsilon$. For a smaller $\epsilon$ the similarity of both asymptotic series is expected to improve due to a wider interval of integration. However, the value $x_c$ will be higher and the similarity will start at the rightmost part of the axis.

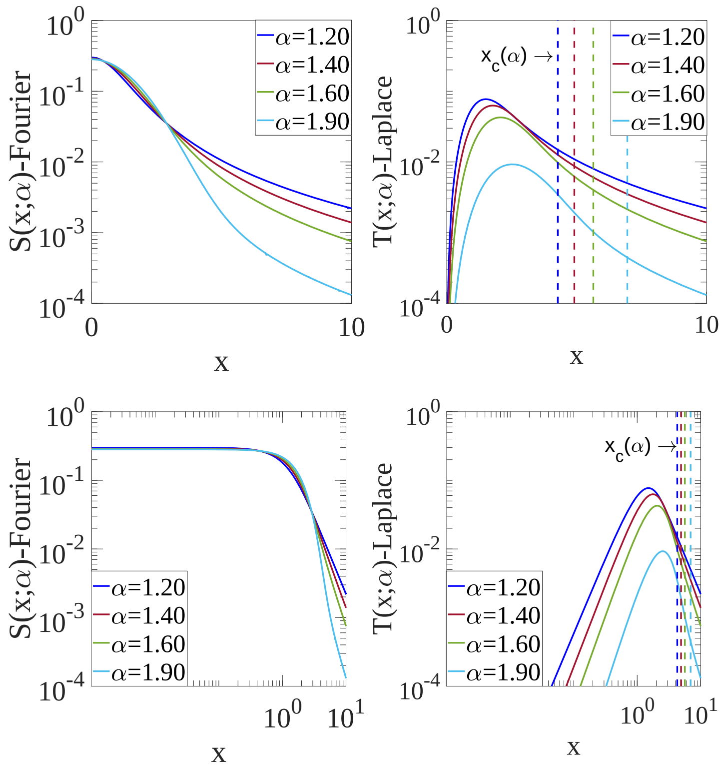


**Figure 5**. *Comparison of numerical integration $1 < \alpha < 2$ between Fourier and Laplace transform of stable $S(x;\alpha)$ and trans-stable $T(x;\alpha)$ functions using recursive adaptive Simpson quadrature method [37]. The absolute error tolerance of the method is $\xi = 3.5 \times 10^{-8}$. Top plots are shown in semi-logarithmic scale and in logarithmic scale.*

## IV. ASYMPTOTIC EXPANSIONS

Asymptotic expansions are developed to obtain closed-form representations for the stable distribution function $S(x,\alpha)$. These expansions are based on the Taylor series of the complex exponential function,

$$e^z \sim \sum_{k=0}^{n} \frac{z^k}{k!} \quad as \quad n \to \infty. \tag{32}$$

Two different asymptotic expansions will be performed. The first one corresponds to the 'inner expansion'. To get this solution the stable distribution function is evaluated by expanding $e^{ixt}$ of Eq. (10) and (21) around $x = 0$. The second one refers to the 'outer expansion', which is the asymptotic series expansion for $x \to \infty$. When $x >> 1$, the oscillations of the integrands in Eq. (10) and (21) are large. Consequently, there are important cancellations due to factor $e^{ixt}$ in the integral. Thus, we focus our integration in the region with the major contribution in the integral, that is around $t = 0$. In consequence, the amplitude of the integral $e^{t^\alpha}$ is replaced by its Taylor expansion around $t = 0$. To guarantee the convergence of the series expansion, the improper integrals are truncated. The truncation occurs because of the sufficient conditions for Riemann integral existence. These conditions are that the integrand must be bounded and the domain of integration is a closed interval [38, 39].

### A. Inner Expansion

The inner expansion is obtained making a substitution of $e^{ixt}$ by its Taylor series expansion given by Eq. (32) in the integrand of the stable distribution $I$. After this substitution, the integrals in Eq. (10) and (21) can be analytically solved. The difference between these two equations are the truncation on the interval of integration.

For $\alpha \leq 1$ the convergence of the series is slow, demanding a large value of order $n$ in Eq. (32) to reach an acceptable similarity with the original integrand $I$. For this reason, the improper integral is truncated after a small enough amplitude of $I$ is obtained. For $\alpha > 1$ the convergence occurs faster and truncation is not needed.

#### 1. *For $0 < \alpha \leq 1$*

The inner expansion is obtained by substituting $e^{ixt}$ in Eq. (21) by its Taylor expansion using Eq. (32). Then:

$$S_i(x;\alpha,\epsilon) = \frac{1}{\pi} \int_0^{\tau_2(x,\epsilon)} e^{-t^\alpha} e^{ixt} dt \sim \frac{1}{\pi} \int_0^{\tau_2(x,\epsilon)} I_n dt \quad as \quad n \to \infty, \tag{33}$$

where $I_n$ is given by:

$$I_n(x;t,\alpha) = \sum_{k=0}^{n} e^{-t^\alpha} \frac{(ixt)^k}{k!}. \tag{34}$$

The upper limit $\tau_2$ is given by the following equation:

$$\tau_2(x,\epsilon) = -\frac{ln(\epsilon)}{x}. \tag{35}$$

This truncation results from the equation $e^{ix\tau_2} = \epsilon$, where $\epsilon$ represents the tolerance that needs to be small to ensure a cut-off when negligible quantities of $|I|$ and $|I_n|$ are obtained. Consequently, the area under the curve of both functions are similar.

The convergence of $I_n$ to $I$ demands a large value of order $n$ in Eq. (32), as it can be observed in Fig (6). This occurs because of slow decay of $e^{-t^\alpha}$ value for $\alpha < 1$. This is the reason to evaluate the integral in the closed interval $[0,\tau_2]$, where the original integrand $I$ and its Taylor series approximation $I_n$ are similar.

The integrals in Eq. (33) can be solved without difficulty. Then, the inner expansion $s_i$ is given by the real part of this solution,

$$s_i(x;\alpha,\epsilon) = Re(S_i(x;\alpha,\epsilon)).$$

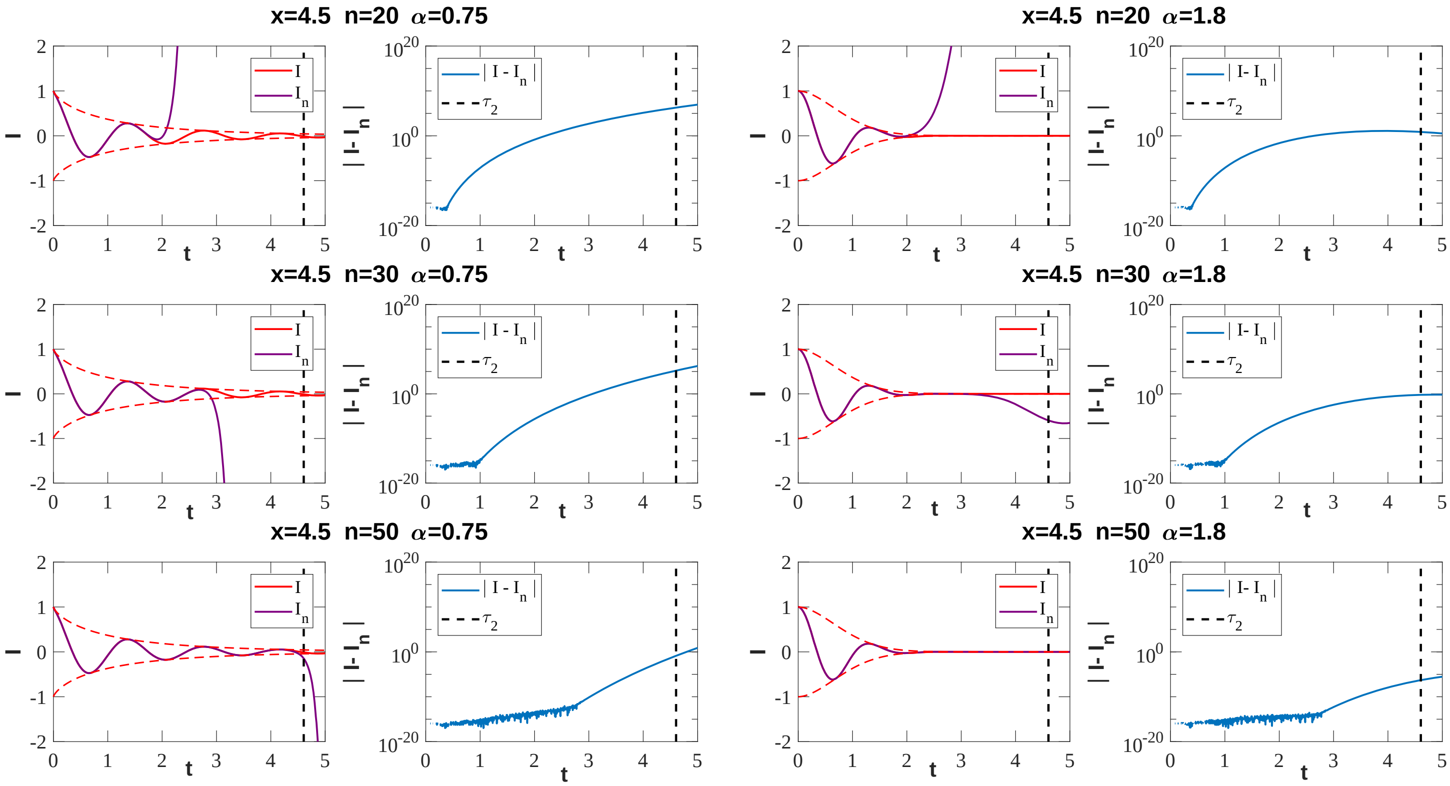


**Figure 6**. *Comparison of $I = e^{t^\alpha} e^{ixt}$ and $I_n = e^{t^\alpha} \sum_{k=0}^{n} \frac{(ixt)^k}{k!}$ of Eq. (33) and (38), where $I_n$ is obtained by replacing $e^{ixt}$ in $I$ by its Taylor expansion. The plots are for $\alpha = 0.75$ (left) and $\alpha = 1.8$ (right). For $\alpha \leq 1$, the truncation is required to ensure a cut-off when negligible quantities of $|I|$ and $|I_n|$ are obtained. The truncated error of the Taylor expansion is measured by using the absolute value of the difference $|I - I_n|$. For $\alpha > 1$, since the convergence is fast the truncation is unnecessary. For these particular examples, the integrand $I$ is evaluated at $x = 4.5$ for three cases of $n = 20, 30, 50$ with $\epsilon = 10^{-9}$.*

Consequently,

$$s_i(x;\alpha,\epsilon) = \frac{1}{\pi\alpha} \sum_{k=0}^{\infty} \frac{x^k}{k!} \gamma\left(\frac{k+1}{\alpha}, \tau_2(x;\epsilon)^\alpha\right) \cos\left(\frac{\pi k}{2}\right), \tag{36}$$

where $\gamma$ represents the incomplete gamma function [40],

$$\gamma(z,b) = \int_0^b x^{z-1} e^{-x} dx. \tag{37}$$

Due to a computation of the incomplete gamma function $\gamma$, Eq. (36) was modified for numerical analysis in Matlab[2] [41].

[2]Note: Matlab defines the incomplete gamma function as $\gamma^*$

$$\gamma^*(b,z) = \frac{1}{\Gamma(z)} \int_0^b x^{z-1} e^{-x} dx$$

where $\Gamma(z)$ is the gamma function.

### 2. *For $1 < \alpha < 2$*

Here it is derived the inner expansion $s_i$ for $\alpha > 1$ from the non-truncated form of stable distribution function. This derivation is made by substituting $e^{ixt}$ in Eq. (10) by its Taylor expansion in Eq. (32), then:

$$S_i(x;\alpha) = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha} e^{ixt} dt \sim \frac{1}{\pi} \int_0^\infty I_n dt \quad as \quad n \to \infty. \tag{38}$$

For $\alpha > 1$, the convergence of integrand $I$ and the integrand after the substitution $I_n$ occurs faster than for $\alpha < 1$. This feature is observed in Figure (6), where an acceptable convergence between $I$ and $I_n$ is obtained with a small $n$ value. Consequently, the integral is evaluated without truncation or taking the limit $\epsilon \to 0$ in Eq. (33).

Then, it is only considered the real part of the solution of Eq. (38),

$$s_i(x;\alpha) = Re(S_i(x;\alpha)).$$

Consequently,

$$s_i(x;\alpha) = \frac{1}{\pi\alpha}\sum_{k=0}^{\infty}\frac{x^k}{k!}\Gamma\left(\frac{k+1}{\alpha}\right)\cos\left(\frac{\pi k}{2}\right), \tag{39}$$

where $\Gamma$ represents the gamma function [40],

$$\Gamma(b) = \int_0^{\infty} x^{b-1}e^{-x}dx.$$

Examples for $\alpha = 0.75$ and $\alpha = 1.80$ are shown on Figures 7 and 8 respectively. In Figure 7, for $\alpha \leq 1$ the truncation $\tau_2$ is needed, otherwise the convergence to stable distribution function will be ultraslow as $n \to \infty$. This is evident when a comparison is made between subfigure 7a and 7b. They represent a non-truncated and truncated stable solution respectively. The subfigure 7b displays an acceptable convergence with a smaller order $n$. In Figure 8, for $\alpha > 1$ the convergence to the stable distribution function occurs faster and no truncation is needed. For both cases the inner expansion $s_i$ behaves well because it converges to $s(x;\alpha)$.

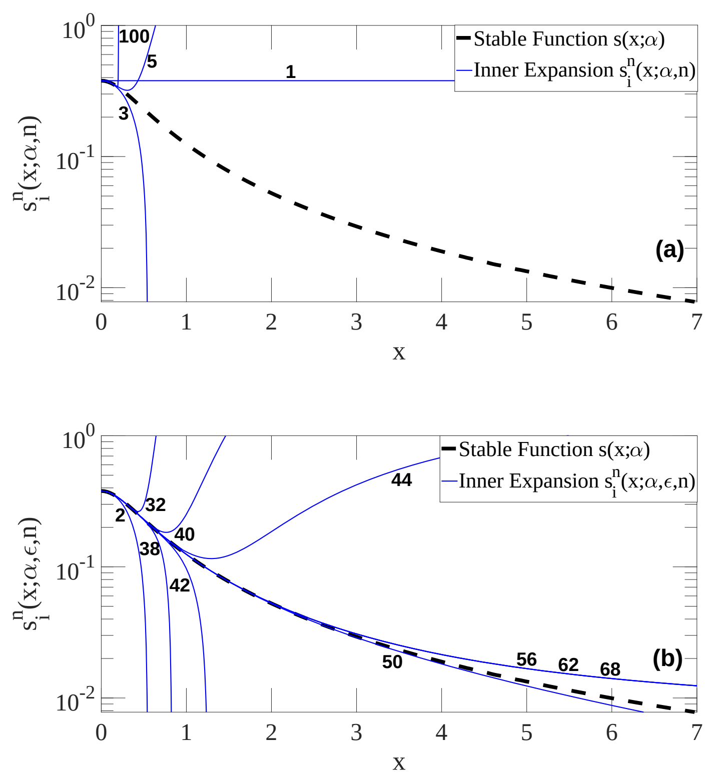


**Figure 7**. *Inner expansion of the stable distribution function for $\alpha = 0.75$. This is obtained by applying Taylor expansion around $t = 0$. The subfigure (a) is a non-truncated integral. The subfigure (b) is the truncated integral with tolerance $\epsilon = 10^{-9}$ in Eq. (36), which displays a fast convergence due to integral's truncation.*

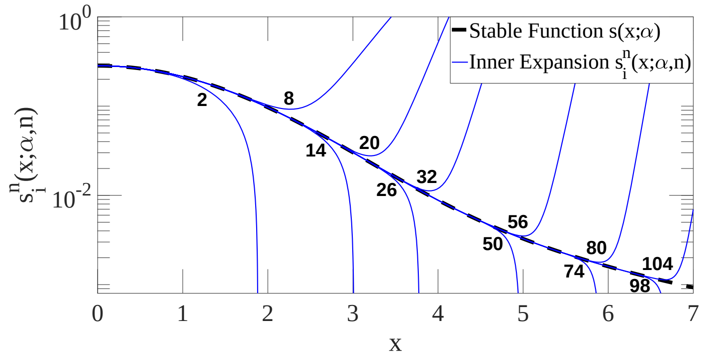


**Figure 8**. *Inner expansion of the stable distribution for $\alpha = 1.80$ as a result of applying a Taylor expansion in the 'exponential of the phase' of the integrand. The figure illustrates the non-truncated stable solution in Eq. (39). This figure displays a fast convergence so that no truncation is needed.*

## B. Outer Expansion

The outer expansion is obtained making a substitution of the amplitude $e^{-t^{\alpha}}$ in the integrand of the truncated stable distribution function $I$ in Eq. (21) by its Taylor series expansion around $t = 0$. Then, the following relation is obtained:

$$S_o(x;\alpha,\epsilon) = \frac{1}{\pi}\int_0^{\tau_3(\epsilon)} e^{-t^{\alpha}}e^{ixt}dt \sim \frac{1}{\pi}\int_0^{\tau_3(\epsilon)} G_n dt \quad as \quad n \to \infty, \tag{40}$$

where $G_n$ is given by:

$$G_n(x;\alpha) = \sum_{k=0}^{n}\frac{(-t^{\alpha})^k}{k!}e^{ixt}. \tag{41}$$

The upper limit $\tau_3$ is given by the following equation:

$$\tau_3(\epsilon) = [-\ln(\epsilon)]^{1/\alpha}. \tag{42}$$

This truncation is calculated from $e^{-\tau_3{}^{\alpha}} = \epsilon$, where $\epsilon$ is defined as tolerance and represents a negligible instantaneous amplitude when $\epsilon$ is small. The truncation allows a faster convergence of $G_n$ to $I$ and reduces the error of integration due to an accurate approximation on the interval $[0, \tau_3]$. The original integrand $I$ and the new integrand after applying Taylor series $G_n$ in Eq. (40) were evaluated in Figure 9. Since the convergence of $G_n$ to $I$ is slow, the truncation $\tau_3$ is considered to define the new interval of integration $[0, \tau_3]$.

To obtain the outer solution $s_o$ a change of variable after the series expansion is applied in Eq. (40). The change of variable is $-u = ixt$, so $-du = ixdt$. This gives us an approximation of the form:

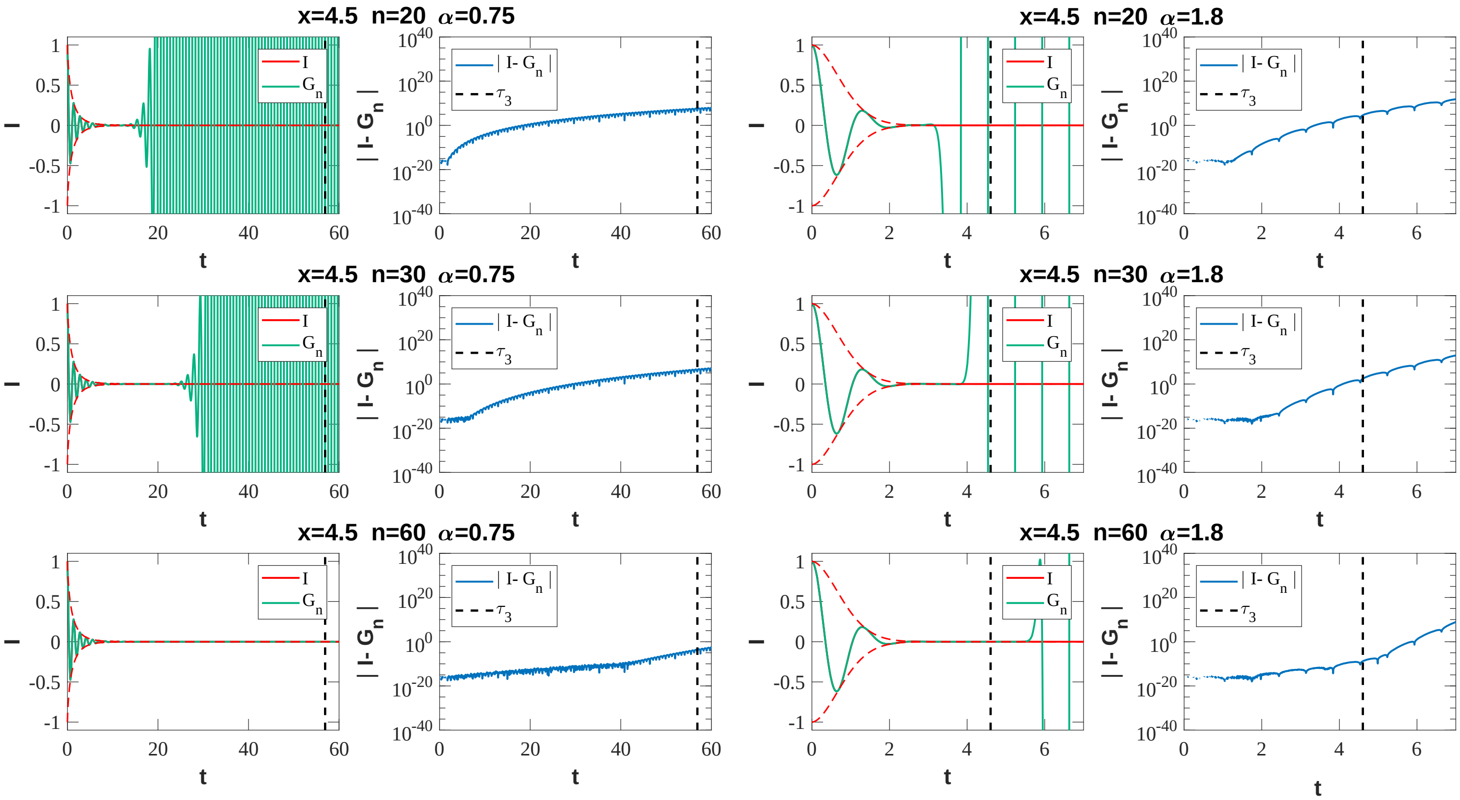


**Figure 9**. *Comparison of $I = e^{t^{\alpha}} e^{ixt}$ and $G_n = \frac{(-t^{\alpha})^k}{k!} e^{ixt} dt$ of Eq. (40), where $G_n$ is the Taylor expansion of $I$ around $t = 0$. The plots correspond to $\alpha = 0.75$ (left) and $\alpha = 1.8$ (right). Truncation of the integral is required for both cases. The reason of that is to reach an accurate approximation between the original integrand $I$ and the one after the series expansion $G_n$. The error is measured by the absolute value of the difference $|I - G_n|$. For these particular examples, the integrand $I$ is evaluated at $x = 4.5$ for three cases of $n = 20, 30, 50$ with $\epsilon = 10^{-9}$.*

$$S_o(x;\alpha,\epsilon) \sim \frac{1}{\pi} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left(\frac{-1}{ix}\right)^{k\alpha+1} \int_0^{-ix\tau_3(\epsilon)} u^{k\alpha} e^{-u} du. \tag{43}$$

To solve the integral, the incomplete gamma function of imaginary argument $\gamma(v, iz)$ is used [40, 42]. The following solution is presented by Barak as a special case of confluent hypergeometric function [42],

$$\gamma(v, iz) = \int_0^{iz} t^{v-1} e^{-t} dt \tag{44}$$

$$= (iz)^v v^{-1} {}_1F_1(v, 1+v, -iz),$$

where ${}_1F_1(v, 1+v, -iz)$ represents the Confluent Hypergeometric function. Then, comparing Eq. (43) and (44), we obtain the following relation betwen the variables, $v = k\alpha + 1$, $z = -x\tau_3$ and $t = u$.

Finally, the real part of the solution is:

$$s_o(x;\alpha,\epsilon) = Re(S_o(x;\alpha,\epsilon)).$$

Consequenlty,

$$s_o(x;\alpha,\epsilon) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\cos(\pi\alpha k)}{k\alpha+1}\right) \quad ...$$

$$...(-\tau_3(\epsilon))^{k\alpha+1} {}_1F_1(k\alpha+1, k\alpha+2, ix\tau_3(\epsilon)). \tag{45}$$

Figure 10 shows the calculation of Eq. 45 for $\alpha = 1.8$. In this figure is evident that the outer expansion $s_o$ converges slowly. This occurs due to computation of the confluent hypergeometric function ${}_1F_1$ which demands considerable computational time. The series that define the function ${}_1F_1$ do not have a trivial structure, this creates numerical issues which makes the calculation computational inefficient [43]. The approximation in Figure 10 shows how the convergence demands a large value of order $n$ to obtain an accurate approximation at the tail. The convergence resembles waves that slowly start to decrease from the tails to the peak of stable distribution. The series until $n = 30$ does not show an acceptable approximation. Only an approximation on tails are obtained after $n = 40$. For $x \to 0$, the series of $s_o$ converges to a specific value different of the stable distribution function. The convergence to the stable distribution function is observed only for $x \to \infty$.

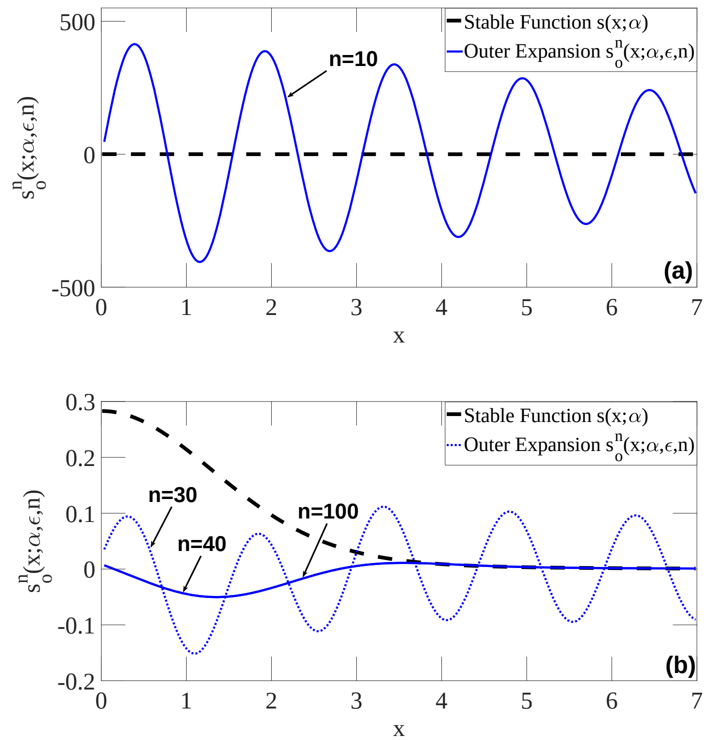


**Figure 10**. *Outer expansion $s_o$ for $\alpha = 1.80$ with tolerance $\epsilon = 10^{-6}$ in Eq. (45). The subfigure (a) and (b) correspond to $n = 10, 30, 40, 100$ respectively.*

### C. Outer expansion by Trans-Stable distribution

Because of the slow convergence of the outer expansion $s_o$ and its wave-like behavior, an alternative approximation is obtained using the trans-stable function $T(x;\alpha)$. As it was previously explained in section III, the solutions of trans-stable $T(x;\alpha)$ and stable $S(x;\alpha)$ functions are identical for $0 < \alpha \le 1$ and similar for $1 < \alpha < 2$ after the $x_c$ value. Consequently, the improper integral in Eq. (19) is used to calculate the series expansions for $0 < \alpha \le 1$ and the truncated trans-stable integral in its Laplace representation in Eq. (24) for $1 < \alpha < 2$.

This outer expansion $t_o$ is given by the analytical solution of the trans-stable function after applying the Taylor series of $e^{-(it)^\alpha}$ around $t = 0$ in the trans-stable integrand $\bar{I}$ using Eq. (32) and (19). Then, the following equation is shown:

$$T_o(x;\alpha,\epsilon) = \frac{1}{\pi}\int_0^{\tau_1(x,\epsilon)} e^{-(it)^\alpha} e^{-xt} i dt \sim \frac{1}{\pi}\int_0^{\tau_1(x,\epsilon)} K_n dt, \tag{46}$$

where $K_n$ is given by:

$$K_n(x) = \sum_{k=0}^{n} \frac{\left(-(it)^\alpha\right)^k}{k!} e^{-xt} i. \tag{47}$$

Due to a slow convergence of $K_n$ to $\bar{I}$ the cut-off $\tau_1$ is applied. The truncation $\tau_1$ has two different expressions. For $\alpha \le 1$, the truncation $\tau_1$ depends on the tolerance $\epsilon$ and for $\alpha > 1$ it depends on the tolerance $\epsilon$ and $x$ values. These expressions will be explained in the following subsections.

To solve the integral in Eq. (46), the following change of variable is applied: $xt = u$ and $xdt = du$. This leads to the following series expansion:

$$T_o(x;\alpha,\epsilon) \sim \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{-1}{ix}\right)^{k\alpha+1} \int_0^{x\tau_1(x,\epsilon)} u^{(k\alpha+1)-1} e^{-u} du. \tag{48}$$

The upper limit of the integral changes from $\tau_1$ to $x\tau_1$, but still remains on the real axis. The integral above can be solved using the incomplete gamma function defined in Eq. (37). Then, the real part of the result is obtained,

$$t_o(x;\alpha,\epsilon) = Re(T_o(x;\alpha,\epsilon)).$$

Consequently,

$$t_o(x;\alpha,\epsilon) = \\ -\frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\left(\frac{1}{x}\right)^{k\alpha+1} \sin\left(\frac{\pi\alpha k}{2}\right)\gamma\left(k\alpha+1, x\tau_1(x,\epsilon)\right). \tag{49}$$

The determination of $\tau_1$ for $\alpha \le 1$ and $\alpha > 1$ is presented in the following subsections.

#### 1. For $0 < \alpha \le 1$

For $\alpha \le 1$, the cut-off $\tau_1$ in Eq. (49) is given by the following equation:

$$\tau_1(\epsilon) = [-\ln(\epsilon)]^{1/\alpha} \quad for \quad \alpha \le 1. \tag{50}$$

This truncation is obtained from $e^{-\tau_1{}^\alpha} = \epsilon$, where the tolerance $\epsilon$ represents a negligible instantaneous amplitude for the integrands in Eq. (46).

#### 2. For $1 < \alpha < 2$

The truncation $\tau_1$ in Eq. (49) for $1 < \alpha < 2$ was already obtained in subsection III-2 and defined by Eq. (30) as:

$$\tau_1(x,\epsilon) = \begin{cases} t_c(\epsilon) & if \quad x > x_c, \\ x/\alpha & if \quad x < x_c, \end{cases} \quad for \quad \alpha > 1,$$

where $t_c$ and $x_c$ were defined by Eq.(31). As indicated in subsection III-2, the value of $\tau_1$ is used to minimize the truncation error and at the same time to make the domain of integration as small as possible.

The outer expansion by the trans-stable function converges to the original trans-stable function. Examples are shown in Figure 11 for $\alpha \leq 1$ and Figure 12 for $\alpha > 1$. Note that in both cases the truncation $\tau_1$ allows a faster and more accurate convergence to the real part of the trans-stable distribution $t(x;\alpha)$. Consequently the outer solution $t_o$ shows an identical solution as $s(x;\alpha)$ for $\alpha \leq 1$ and the same asymptotic behaviour for $\alpha > 1$. For a smaller $\epsilon$ the convergence of these outer expansions to the trans-stable function will occur faster. Also in Figure 11 the non-truncated trans-stable expansion is shown as an expansion that converges extremely slowly requiring a higher order $n$ than truncated trans-stable expansion to obtain an acceptable convergence. In Figure 12 the non-truncated trans-stable expansion does not converge to trans-stable function at all.

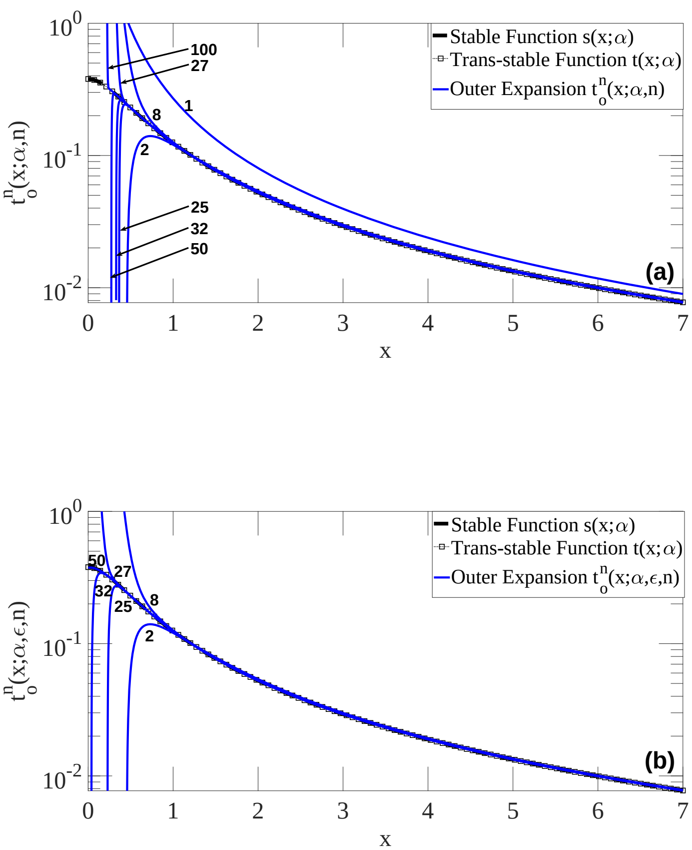


**Figure 11**. *Outer expansion of the trans-stable function for $\alpha = 0.75$. This result is obtained from the Taylor expansion of the integrand around $t = 0$ in Eq. (49) and (50). The subfigure (a) is the non-truncated integral that shows slow convergence. The subfigure (b) corresponds to truncated integral with tolerance $\epsilon = 10^{-6}$. The subfigure (b) displays a faster convergence to the trans-stable function as a result of the truncation of the integral.*

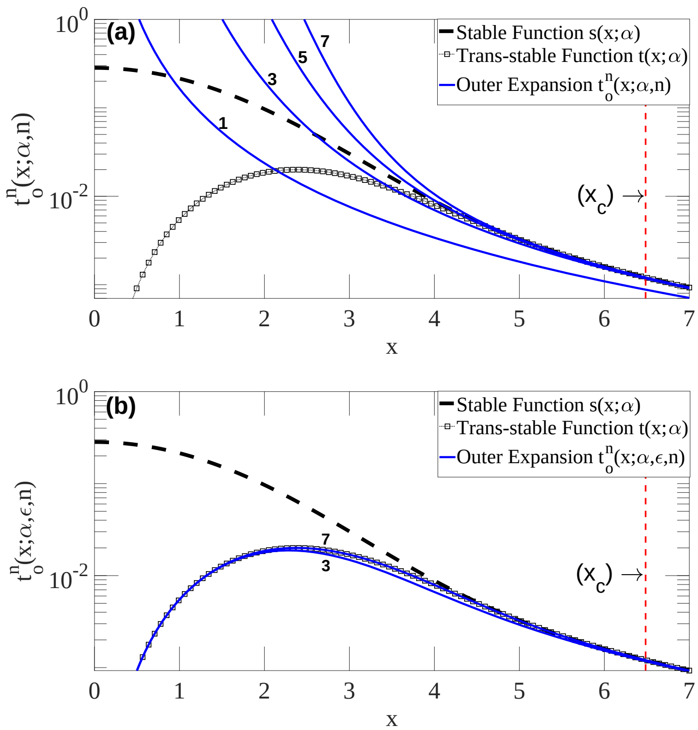


**Figure 12**. *Outer expansion of the trans-stable for $\alpha = 1.80$ as a result of applying Taylor expansion of the integrand around $t = 0$ in Eq. (30) and (49). The subfigure (a) shows that the non-truncated integral does not converge to the trans-stable function. The subfigure (b) corresponds to the truncated integral with tolerance $\epsilon = 10^{-6}$. The subfigure (b) displays a fast convergence as a result of the truncation of the integral.*

## V. UNIFORM SOLUTION

The uniform solution is presented as the combination of the inner solution and the outer solution to construct an approximation valid for all $x \in [-\infty, \infty]$. To construct the uniform solution an asymptotic matching method based on boundary-layer theory is applied [44, 45]. This method is based on superposing the inner and outer solution and substracting the overlap between them,

$$s_u(x) = y_{out}(x) + y_{in}(x) - y_{overlap}(x). \qquad (51)$$

The overlap is defined as the limit of the rightmost edge of $y_{in}$ and the leftmost edge of $y_{out}$,

$$y_{overlap} = \lim_{x\to 0} y_{out} = \lim_{x\to\infty} y_{in}. \qquad (52)$$

For this case, our proposed uniform solution $s_u$ is constructed based on our inner expansion $s_i$ and our outer expansion $t_o$. These previous solutions were already defined in section IV.

For a better understanding of our uniform solution $s_u$, two sub-sections A and B are presented. Sub-section A

<table>
<tr><td>Range of $\alpha$</td><td>$0 < \alpha \leq 1$</td><td>$1 < \alpha < 2$</td></tr>
<tr><td>Normalized stable distribution ($s$)</td><td colspan="2">$s(x;\alpha) = \frac{1}{\pi} \int\limits_0^\infty e^{-t^\alpha} \cos(tx)dt$</td></tr>
<tr><td>Normalized trans-stable distribution ($t$)</td><td colspan="2">$t(x;\alpha) = \frac{1}{\pi} \int\limits_0^\infty e^{-t^\alpha \cos(\frac{\pi\alpha}{2}) - xt} \sin(t^\alpha \sin(\frac{\pi\alpha}{2}))dt$</td></tr>
<tr><td>Inner expansion ($s_i^n$)</td><td>$s_i^n(x;\alpha,\epsilon) = \frac{1}{\pi\alpha} \sum_{k=0}^{n} \frac{x^k}{k!} \gamma\left(\frac{k+1}{\alpha}, {\tau_2}^\alpha\right) \cos\left(\frac{\pi k}{2}\right)$<br>$\tau_2 = \frac{-\ln(\epsilon)}{x}$</td><td>$s_i^n(x;\alpha) = \frac{1}{\pi\alpha} \sum_{k=0}^{n} \frac{x^k}{k!} \Gamma\left(\frac{k+1}{\alpha}\right) \cos\left(\frac{\pi k}{2}\right)$</td></tr>
<tr><td>Outer expansion ($s_o^n$)</td><td colspan="2">$s_o^n(x;\alpha,\epsilon) = -\frac{1}{\pi} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left(\frac{\cos(\pi\alpha k)}{k\alpha+1}\right) \quad (-\tau_3)^{k\alpha+1} {}_1F_1(k\alpha+1, k\alpha+2, ix\tau_3)$<br>$\tau_3 = [-\ln(\epsilon)]^{1/\alpha}$</td></tr>
<tr><td rowspan="2">Outer solution [3] ($t_o^n$)</td><td colspan="2">$t_o^n(x;\alpha,\epsilon) = -\frac{1}{\pi} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left(\frac{1}{x}\right)^{k\alpha+1} \gamma\left(k\alpha+1, x\tau_1\right) \sin\left(\frac{\pi\alpha k}{2}\right)$</td></tr>
<tr><td>$\tau_1 = (-\ln(\epsilon))^{1/\alpha}$</td><td>$\tau_1 = \begin{cases} t_c & if \quad x > x_c \\ x/\alpha & if \quad x < x_c \end{cases}$</td></tr>
<tr><td>Complete and incomplete gamma functions ($\Gamma$) & ($\gamma$)</td><td colspan="2">$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \quad \gamma(z,b) = \int_0^b x^{z-1} e^{-x} dx$</td></tr>
</table>

**Table II**. Summary of Inner and Outer Solutions

contains a summary of inner and outer expansions previously obtained. In sub-section B the steps taken to obtain $s_u$ are explained.

### A. Summary of inner and outer expansions

Table II contains the normalized stable and trans-stable distribution and the summary of previous results obtained from stable and trans-stable functions by applying Taylor expansions. The series refers to one inner expansion $s_i$ and two outer expansions $s_o$ and $t_o$.

[3]Note: Refer to equation Eq. (30) to obtain $t_c$ and $x_c$ value for $1 < \alpha < 2$

For the inner expansion $s_i$, the solution for $\alpha \leq 1$ corresponds to a truncated stable solution which allows a faster convergence. For $\alpha > 1$ the series is obtained from the non-truncated stable solution. The only difference between them is the use of the incomplete gamma function $\gamma$ in the solution for $\alpha \leq 1$, where $\Gamma(z) = \lim_{b\to\infty} \gamma(z,b)$. Consequently, for both cases the truncated series can provide a good approximation. However, in the case of $\alpha \leq 1$ we must take the limit as:

$$s(x;\alpha) = \lim_{\epsilon\to 0} \left( \lim_{n\to\infty} s_i^n(x;\alpha,\epsilon) \right) \quad for \quad x < \infty.$$

In general the order how we apply the limits cannot be exchanged. However, in the case of $\alpha > 1$ the order of the limits does not affect the convergence. Taking a small value of $\epsilon$ ensures a faster convergence.

For the outer expansion two expressions were derived.

The first outer espansion $s_o$ is obtained by performing the Taylor expansion around $t = 0$ on the truncated stable distribution. This solution displays a slow convergence for $n \to \infty$. The second outer expansion $t_o$ is obtained by applying the Taylor expansion on the truncated trans-stable function for $x \to \infty$. The truncation of $t_o$ depends on $\alpha$ and there are two different cases. For $\alpha \leq 1$ it converges to the exact solution of $s(x;\alpha)$ and for $\alpha > 1$ it converges to the same solution at the tails of $s(x;\alpha)$. To guarantee convergence, we need to take the limit as,

$$t(x;\alpha) = \lim_{\epsilon\to 0}\left(\lim_{n\to\infty} t_o^n(x;\alpha,\epsilon)\right) \quad for \quad x > 0.$$

Exchanging the order of the limits will affect the convergence. The outer expansion that will be used is $t_o$, because it displays a faster convergence and it does not exhibit wavelike behavior.

## B. Steps to obtain the uniform solution

To obtain the uniform solution $s_u$ the condition in Eq. (52) needs to be satisfied. The inner expansion $s_i$ and the outer expansion $t_o$ have to be multiplied with an appropriate coefficient $A(x)$ to obtain the asymptotic solutions with a common matching value $y_m$. These operations will allow us to obtain $y_{out}$ and $y_{in}$. Consequently, Eq. (51) will be applied to obtain the closed-form solution of the stable distribution function.

Below the steps are explained to obtain the location of the matching between the inner and the outer solutions $(x_m, y_m)$, the coefficient $A(x)$, and the uniform solution $s_u$.

### 1. *Finding inner and outer limit* $(x_m, y_m)$

Considering $s_i$ and $t_o$ as good approximations to the stable distribution function, we must require that the inner and the outer expansions will be close enough before matching them [46]. Consequently, the point where the matching between $s_i$ and $t_o$ takes place is $(x_m, y_m)$ and it represents the location where the minimal vertical distance between the inner $s_i$ and the outer solution $t_o$ occurs.

The distance function between $s_i$ and $s$ is defined as $\delta_i$ and the distance function between $t_o$ and $s$ is $\delta_o$. Consequently, $(x_m, y_m)$ is the point where the Pythagorean addition of these distances is minimal Eq. (53):

$${\delta_i}^2(x;\alpha,\epsilon) = (s(x;\alpha) - s_i(x;\alpha,\epsilon))^2,$$

$${\delta_o}^2(x;\alpha,\epsilon) = (s(x;\alpha) - t_o(x;\alpha,\epsilon))^2,$$

$$\delta^2(x;\alpha,\epsilon) = {\delta_o}^2(x;\alpha,\epsilon) + {\delta_i}^2(x;\alpha,\epsilon),$$

$$\left.\frac{d\left(\delta^2(x;\alpha,\epsilon)\right)}{dx}\right|_{x_m} = 0. \tag{53}$$

The $x_m$ value is obtained from the previous equation. Then, $y_m$ is defined by the equidistant point between both functions,

$$y_m = \frac{s_i(x_m) + t_o(x_m)}{2}. \tag{54}$$

### 2. *Defining the inner and the outer solutions* $y_{in}$ *and* $y_{out}$

To obtain the uniform solution $s_u$, the asymptotic matching method based on boundary layer theorem [44] is applied. Consequenlty, the inner solution $y_{in}$ and the outer solution $y_{out}$ must have a matching asymptotic behaviour. More precisely, the limit of the outer solution $y_{out}$ when $x \to 0$ should correspond to the limit of the inner solution $y_{in}$ when $x \to \infty$. To obtain $y_{in}$ and $y_{out}$ solutions, the series expansions $s_i$ and $t_o$ are multiplied by an appropiate coefficients to meet the requirements of matching asymptotic expansions, so the $y_{in}$ and $y_{out}$ are defined as follows:

$$y_{in}(x;\alpha,\epsilon,\mu) = (s_i(x) - y_m)\,(1 - A(x;\mu)) + y_m, \tag{55}$$

$$y_{out}(x;\alpha,\epsilon,\mu) = (t_o(x) - y_m)A(x;\mu) + y_m, \tag{56}$$

where the overlapping factor $A(x)$ is defined as:

$$A(x;\mu) = \frac{1}{2}\left(1 + \tanh\left(\frac{x - x_m}{\mu}\right)\right). \tag{57}$$

The $A(x;\mu)$ is used to smooth $s_i$ and $t_o$ and provides them with a symmetric overlap section around $x_m$ and gives $y_{in}$ and $y_{out}$ an asymptotic behaviour. The variable $\mu$ determines the width of the overlap between $y_{in}$ and $y_{out}$.

It is easy to see that Eq. (55) and Eq. (56) satisfy Eq. (52), where the limits of $y_{out}$ and $y_{in}$ converge to a constant value $y_m$.

### 3. *Defining the Uniform Solution* $s_u$

The inner solution $y_{in}$ Eq.(55) and the outer solution $y_{out}$ in Eq.(56) were defined to fulfill the requirements for matching asymptotic expansions. Then, Eq.(51) is applied to obtain the uniform solution $s_u$,

$$s_u^{n_i,n_o}(x;\alpha,\epsilon,\mu) = \frac{t_o^{n_o}(x;\alpha,\epsilon)}{2} + \frac{s_i^{n_i}(x;\alpha,\epsilon)}{2}.... \\ ....+\tanh\left(\frac{x - x_m}{\mu}\right)\left(\frac{t_o^{n_o}(x;\alpha,\epsilon)}{2} - \frac{s_i^{n_i}(x;\alpha,\epsilon)}{2}\right). \tag{58}$$

#### 4. *Find the best $s_u$ by choosing the most appropiate $\mu$*

The width of the overlap between $y_{in}$ and $y_{out}$ can be optimized to obtain the closest solution $s_u$ of the stable distribution function. The most appropriate value of $\mu$ needs to be obtained for each particular value of $\alpha$. For that, the least square method will be applied between the original $s(x;\alpha)$ and the new closest solution $s_u(x;\alpha,\epsilon,\mu)$. Applying Eq. (7) and (58) the following equation is obtained:

$$L(\mu) = \sum_{i=1}^{N} \left(s_u(x_i;\alpha,\epsilon,\mu) - s(x_i;\alpha)\right)^2, \tag{59}$$

where the $N$ value represents the length of the sample used to minimize L.

The similarity between the exact solution of $s(x;\alpha)$ and the uniform solution $s_u(x;\alpha,\epsilon,\mu)$ is observed in Figure 13 and 14 for $\alpha = 0.75$ and $\alpha = 1.80$ respectively. For $\alpha < 1$, a good approximation between $s(x;\alpha)$ and $s_u(x;\alpha,\epsilon,\mu)$ is obtained in the tails after mixing two different orders. The order for the inner solution is $n_i = 6$, which makes the solution concave upward. The order for the outer solution is $n_o = 17$, which makes the solution concave downward. This combination of orders will ensure a good matching asymptotic behaviour. For $\alpha > 1$, the uniform solution works well, and a good uniform solution is obtained quickly with a lower order $n = 6$.

Lower orders can be used for both cases, where the most important aspect to consider is the different concavity between $y_{in}$ and $y_{out}$ for the matching asymptotic behaviour. The concavity of the inner and outer solution is defined by the trigonometric element in each solution.

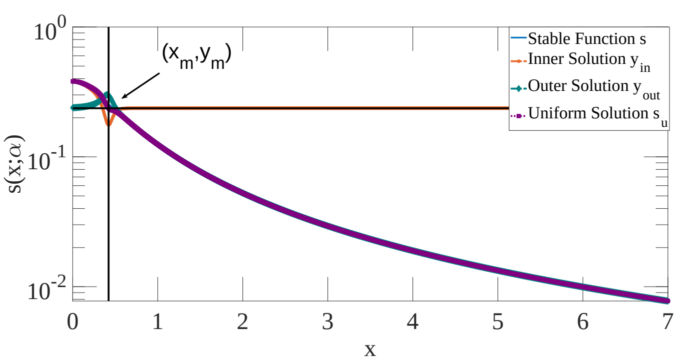


**Figure 13**. *Uniform solution $s_u$ for $\alpha = 0.75$ as a result of joining the inner solution $y_{in}$ with the outer solution $y_{out}$. The tolerance $\epsilon = 10^{-6}$, $\mu = 0.052$ and $n_i = 6$ and $n_o = 17$.*

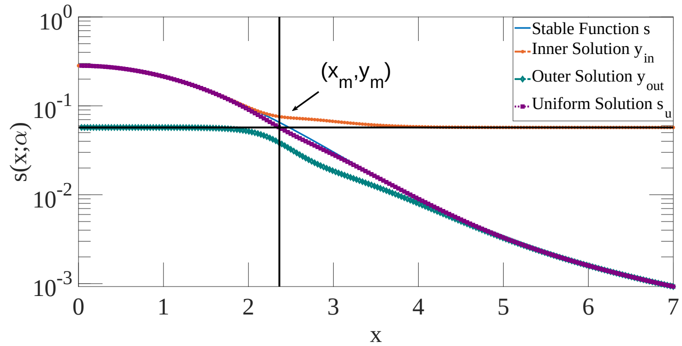


**Figure 14**. *Uniform solution $s_u$ for $\alpha = 1.80$ as a result of joining the inner solution $y_{in}$ with the outer solution $y_{out}$. The tolerance $\epsilon = 10^{-6}$, $\mu = 0.4$ and $n = 8$.*

## VI. CONCLUSIONS

In this paper we presented a uniform solution of the stable distribution. This solution converges to the stable distribution function in the full range of $x$ values $-\infty < x < \infty$. This condition makes our uniform solution more robust than previous analytical expressions that were only applicable for extreme values $x \to 0$ or $x \to \infty$. Also, our uniform solution removes the negative values obtained in previous numerical solutions of the stable distribution function for all $\alpha$ values, which makes this solution more reliable because a probability density function must be always positive.

The uniform solution is the result of an asymptotic matching between the inner and outer expansions. The inner expansion results from the Taylor series expansion of the characteristic function of the stable distribution around $x = 0$. The outer expansion is obtained from the Taylor expansion of the integrand of the trans-stable function around $t = 0$. The convergence of these expansions is guaranteed if the integrands are truncated, and the speed of convergence depends on how is the truncation implemented.

For $\alpha \leq 1$, the uniform solution provides a good approximation for all the range of $x$ values. Also, the numerical integration of the trans-stable function constitutes a second option which allows us to obtain a robust numerical solution of the stable distribution function and removes the oscillations. For $\alpha > 1$, the uniform solution provides an analytical solution of the stable distribution function based on fast converging series.

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